The geometry of the $SU(2)$ Kepler problem

TOSHIHIRO IWAI
Department of Applied Mathematics and Physics
Kyoto University
Kyoto 606, Japan

Abstract. *The $SU(2)$ Kepler problem is defined and analyzed, which is a Hamiltonian system reduced from the conformal Kepler problem on $T^*(\mathbf{R}^8 - \{0\})$ by the use of the symplectic $SU(2)$ action lifted from the $SU(2)$ left action on the $SU(2)$ bundle $\mathbf{R}^8 - \{0\} \to \mathbf{R}^5 - \{0\}$. This reduced system has a parameter $\mu \in su(2)$ coming from the value of the moment map associated with the symplectic $SU(2)$ action. If $\mu \neq 0$, the phase space of this system have a bundle structure with base space $T^*(\mathbf{R}^5 - \{0\})$ and fibre S^2. The fibre, a (co)adjoint orbit through μ for $SU(2)$, represents the internal degrees of freedom, called the isospin, of the particle of this system. The $SU(2)$ Kepler problem with $\mu \neq 0$ is then interpreted as describing the motion of a classical particle with isospin in the Newtonian potential plus a specific repulsive potential together with a Yang-Mills field. This Yang-Mills field is to be referred to as BPST Yang's monopole field in $\mathbf{R}^5 - \{0\}$, since it becomes the Belavin-Polyakov-Schwartz-Tyupkin instanton, restricted on S^4. If $\mu = 0$, the $SU(2)$ Kepler prblem reduces to the ordinary Kepler problem. Like the ordinary Kepler problem, the Hamiltonian flows of the $SU(2)$ Kepler problem of negative energy are all closed. It is shown in an explicit manner that the energy manifolds and isoenergetic orbit spaces for the $SU(2)$ Kepler problem of negative energy are both homogeneous manifolds on which $SU(4)$ acts transitively to the right; those homogeneous manifold are classified into two, according as the parameter μ is zero or not. For a certain value of μ, however, they contracts to the manifold which represents the set of all the equilibrium states. The isoenergetic orbit spaces are finally shown to be symplectomorphic to certain Kirillov-Kostant-Souriau coadjoint orbits for $U(4)$, if μ is not the exceptional value mentioned above.*

Key-Words: *$SU(2)$ bundle, Hamiltonian G-space, $SU(2)$ Kepler problem, Isoenergetic manifold, Isoenergetic orbit space.*
1980 MSC: 58F05, 53C80.

1. INTRODUCTION

The usual Kepler problem has long been studied from various points of view. From the symmetry point of view, the author and Uwano [1] defined and analyzed the MIC-Kepler problem as a generalization of the Kepler problem, which is defined on the cotangent bundle $T^*(\mathbb{R}^3 - \{0\})$ equipped with a symplectic form other than the standard one. In classical theory, the energy manifold for the MIC-Kepler problem of negative energy was shown to be topologically the same as the «regularized» negative energy manifold for the Kepler problem, and to admit the symmetry group $SO(4) = SU(2) \times SU(2)/\mathbf{Z}_2$. In quantum theory [2], the negative-energy eigenspaces for the quantized MIC-Kepler problem proves to carry all the unitary irreducible representations of $SU(2) \times SU(2)$. In this analysis, the principal bundle $U(1) \to \mathbf{R}^4 - \{0\} \to \mathbb{R}^3 - \{0\}$, an extension of the Hopf bundle $U(1) \to S^3 \to S^2$, played a key role. In celestial mechanics, the map $\mathbf{R}^4 - \{0\} \to \mathbf{R}^3 - \{0\}$ is called the KS (Kustaanheimo-Stiefel) transformation [3], [4].

Cordani [5] pointed out the MIC-Kepler problem is associated with the moment map $\mathbb{C}^{2,2} \to su(2,2)^*$.

Another generalization of the Kepler problem is possible if the principal bundle $SU(2) \to \mathbf{R}^8 - \{0\} \to \mathbf{R}^5 - \{0\}$, an extension of the Hopf bundle $SU(2) \to S^7 \to S^4$, is used. In this paper, a continuation of the previus one [1], the $SU(2)$ Kepler problem is defined and analyzed as a classical mechanical system.

Like the MIC-Kepler problem, the $SU(2)$ Kepler problem is reduced from the conformal Kepler problem [6] defined on the cotangent bundle $T^*(\mathbb{R}^8 - \{0\})$, by using the symplectic action lifted from the action of the structure group $SU(2)$ on the bundle $\mathbf{R}^8 - \{0\} \to \mathbf{R}^5 - \{0\}$. As for the reduction of the cotangent bundle T^*P with P a principal fiber bundle, Montgomery [7] showed what phase space the T^*P is reduced to, and that the reduced phase space is diffeomorphic to the phase space that Sternberg [8] set up for a classical particle in the presence of a Yang-Mills field. The present article is hence in part an application of Montgomery's, but has no concern with Wong's equation which Montgomery treated in his paper as the equation describing the motion of the classical particle in the presence of a Yang-Mills field. The Hamiltonian to be treated in this paper is a reduced one from that of the conformal Kepler problem.

The main interest of this article will center on the geometry of the negative-energy manifolds. Since every orbit of the $SU(2)$ Kepler problem of negative energy proves to be closed, and thereby defines a $U(1)$ action, the isoenergetic orbit spaces are studied as well. The $SU(4)$ symmetry will play a key role in working out questions as to what manifolds the negative-energy manifold and the isoenergetic orbit space are diffeomorphic to, respectively.

The organization of this paper is as follows: Section 2 is a review of the principal bundle $SU(2) \to \mathbf{R}^8 - \{0\} \to \mathbf{R}^5 - \{0\}$ along with its natural connection. The structure group is here supposed to act on $\mathbf{R}^8 - \{0\}$ to the left.

Section 3 is concerned with the moment map $J : T^*(\mathbf{R}^8 - \{0\}) \to su(2)^*$ associated with the lifted action of the structure group $SU(2)$, where $su(2)^*$ is the dual to the Lie algebra $su(2)$ of $SU(2)$. By using J, the reduced phase space is formed and shown to be a fibre bundle over $T^*(\mathbf{R}^5 - \{0\})$ with fibre S^2. This imples that a particle described by the reduced system carries internal degrees of freedom on the fibre.

In Section 4, the $SU(2)$ Kepler problem is defined by reducing the Hamiltonian of the conformal Kepler problem through the reduction procedure discussed in Sec. 3. This system describes the motion of a particle with internal structure in the presence of $BPST$-Yang's monopole field [9], [10] along with a centrifugal potential plus the Newtonian potential. The symplectic form of this system indeed contains the term that depends on the curvature form, which gives $BPST$-Yang's monopole field on $\mathbf{R}^5 - \{0\}$, and becomes the well-known Belavin-Polyakov-Schwartz-Tyupkin ($BPST$) instanton [10], when restricted on S^4.

Section 5 gives negative-energy manifolds of the $SU(2)$ Kepler problem, which are described as homogeneous spaces with $SU(4)$, where $SU(4)$ acts on $T^*(\mathbf{R}^8 - \{0\})$ to the right. It is to be noted here that $SU(2)$ and $SU(4)$ acts on $T^*(\mathbf{R}^8 - \{0\})$ to the left and to the right, respectively, so that the actions commute to each other. According as the value of the moment map J is zero or not, the negative-energy manifold proves to be $SU(4)/SU(2) \times SU(2)$ or $SU(4)/U(1) \times SU(2)$. However, for a certain value of J, the negative-energy manifold becomes the complex projective space $\mathbb{C}P^3$, which is also a homogeneous space $SU(4)/S(U(1) \times U(3))$.

Section 6 deals with the isoenergetic orbit spaces for the $SU(2)$ Kepler problem. As will be shown easily, every Hamiltonian flow of the $SU(2)$ Kepler problem of negative energy is closed, so that one can think of the space of isoenergetic orbits (or flows). This orbit space is also expressible as a homogeneous space with $SU(4)$. According as the value of J is zero or not, it turns out to be $SU(4)/S(U(2) \times U(2))$ or $SU(4)/S(U(1) \times U(1) \times U(2))$. These are shown to be diffeomorphic, respectively, to the complex Grassmann manifold of complex two-planes in $\mathbb{C}^4$ and to the flag manifold consisting of all the pairs $(U_1, U_2), U_1$ being a one-dimensional subspace of $\mathbb{C}^4$, U_2 a two-dimensional subspace containing U_1. For the special value of J mentioned in Sec. 5, however, the isoenergetic orbit space is equal to the negative-energy manifold, since the orbits of the $SU(2)$ Kepler problem defines then the trivial action, viz., the identity.

In Section 7, the isoenergetic orbit spaces discussed in Sec. 6 will be understood as reduced phase spaces associated with the $U(1) \times SU(2)$ left action on $T^*\mathbf{R}^8$, where $U(1)$ and $SU(2)$ come from the closed flows and the structure group, respectively. These spaces are then realized as coadjoint orbits of $U(4)$ by applying the moment map F associated with the $U(4)$ symmetry, where $U(4)$ right action is used, instead of the $SU(4)$ right action discussed in preceeding sections, for the sake of convenience.

The Kirillov-Kostant-Souriau form [11] on the coadjoint orbit is shown to be related with the standard symplectic form on $T^*\mathbf{R}^8$ through the moment map F.

Section 8 has concluding remarks, which are broken up into two. The first part of this section contains comparison of the $SU(2)$ Kepler problem with the Sternberg recipe for a classical particle in a Yang-Mills field [8]. In the latter part, the coadjoint structure discussed in Sec. 7 are generalized to the case where commutative actions of two Lie groups, $H \times K$, are given on a Hamiltonian $H \times K$-space. For $H = U(4)$ and $K = U(1) \times SU(2)$, part of results in Sec. 7 comes out.

Appendix contains explicit expression of settings on the $SU(2)$ bundle $\mathbf{R}^8 - \{0\} \to \mathbf{R}^5 - \{0\}$.

2. THE $SU(2)$ BUNDLE $\mathbf{R}^8 - \mathbf{R}^5$

This section is a brief review of the principal bundle $SU(2) \to \dot{\mathbf{R}}^8 \to \dot{\mathbf{R}}^5$, where $\dot{\mathbf{R}}^8 := \mathbf{R}^8 - \{0\}$ and $\dot{\mathbf{R}}^5 := \mathbf{R}^5 - \{0\}$. We treat the quaternions $\mathbf{H}$ in 2×2 complex matrix form and set

$$\mathbf{H} = \left\{ \begin{pmatrix} x_0 + ix_1 & -x_2 + ix_3 \\ x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} ; x_j \in \mathbf{R},\ j = 0, \ldots, 3 \right\} . \tag{2.1}$$

The inner product and an anti-symmetric form on $\mathbf{H}$ are defined, for $X, Y \in \mathbf{H}$, respectively, by

$$(X|Y) = \frac{1}{2}\mathrm{tr}(XY^*) , \tag{2.2}$$

$$\gamma(X,Y) = \frac{1}{2}(XY^* - YX^*) , \tag{2.3}$$

where the superscript asterisk denotes the Hermitian conjugate. Note that γ takes values in $su(2)$, the imaginary part of $\mathbf{H}$. For $g \in SU(2)$, the subspace of H of unit modulus, these forms are subject to the transformation

$$(gX|gY) = (X|Y) , \tag{2.4a}$$

$$\gamma(gX, gY) = \mathrm{Ad}_g \gamma(X,Y) . \tag{2.4b}$$

The inner product of $su(2)$ is defined, for $\xi, \eta \in su(2)$, by

$$\langle \xi | \eta \rangle = \frac{1}{2}\mathrm{tr}(\xi\eta^*) . \tag{2.5}$$

Further, for $\xi \in su(2)$ and $X, Y \in \mathbf{H}$, one has

$$(X|\xi Y) = \langle \gamma(X,Y) | \xi \rangle . \tag{2.6}$$

Now consider the product space $\mathbf{H}^2$. By $\dot{\mathbf{H}}^2$ we denote the set of all the pairs $(X,Y) \neq (0,0)$. Then $\dot{\mathbf{H}}^2 \cong \dot{\mathbf{R}}^8$. The group $SU(2)$ acts diagonally on $\dot{\mathbf{H}}^2$; for $g \in SU(2)$, one has

$$\Phi_g(X,Y) = (gX, gY)\ . \tag{2.7}$$

The principal fibre bundle

$$SU(2) \to \dot{\mathbf{H}}^2 \to \dot{\mathbf{H}}^2/SU(2) \cong \dot{\mathbf{R}}^5 \tag{2.8}$$

is realized by the projection

$$\pi : (X,Y) \mapsto (2X^*Y, \det X - \det Y) \in \mathbf{H} \times \mathbf{R}\,. \tag{2.9}$$

Setting

$$\begin{cases} 2X^*Y = W, \\ \det X - \det Y = w_4\ , \end{cases} \tag{2.10}$$

one has

$$||w||^2 := \sum_{k=0}^{4} w_k^2 = (|x|^2 + |y|^2)^2\ , \tag{2.11}$$

where X, Y and W have the components $(x_j), (y_j)$ and $(w_j), j = 0, \dots, 3$, respectively, like in (2.1), and

$$|x|^2 := \sum_{j=0}^{3} x_j^2 = \det X, \text{etc}.$$

The explicit form of w_j will be given in Appendix.

We now proceed to the natural connection on $\dot{\mathbf{H}}^2$. Let ξ be a vector in $su(2)$. Then ξ gives rise to a fundamental vector field $R_\xi(q)$, $q = (X,Y) \in \dot{\mathbf{H}}^2$. At every point $q = (X,Y)$ the set of $R_\xi(q)$ with $\xi \in su(2)$ form the vertical subspace of the tangent space $T_q(\dot{\mathbf{H}}^2)$. The horizontal subspace [12] is defined to be the orthogonal complement of the vertical subspace with respect to the standard metric

$$K_q := (dX|dX) + (dY|dY)\ . \tag{2.12}$$

For a certain basis $E_a, a = 1,2,3$ of $su(2)$ (see (A.2) in Appendix), we set $F_a = R_{E_a}(q)$. Then F_a are orthogonal to one another with respect to K. Fortunately, we can extend F_a to an orthogonal system $\{F_a, H_j\}, a = 1,2,3, j = 0,\dots,4$, of vector fields such that at every point $q \in \dot{\mathbb{H}}^2$, H_j form a basis of horizontal subspace of $T_q(\dot{\mathbb{H}}^2)$ (see Appendix for the explicit form of F_a and H_j). For this orthogonal system, one has

$$
\begin{aligned}
&K(F_a, F_b) = (|x|^2 + |y|^2)\delta_{ab}, \qquad a,b = 1,2,3 \\
&K(F_a, H_k) = 0, \qquad\qquad (2.13)\\
&K(H_j, K_k) = (|x|^2 + |y|^2)\delta_{jk}. \qquad j,k = 0,\dots,4 .
\end{aligned}
$$

For the projection π given by (2.9), one has

$$\pi_* H_j = 2\|w\|\partial/\partial w_j, \qquad j = 0,\dots,4, \tag{2.14}$$

so that the horizontal lifts $(\partial/\partial w_j)^*$ of $\partial/\partial w_j$ are expressed as

$$\left(\frac{\partial}{\partial w_j}\right)^* = \frac{1}{2r}H_j, \qquad j = 0,\dots,4, \tag{2.15}$$

with

$$r = \|w\| = |x|^2 + |y|^2 . \tag{2.16}$$

In a dual manner, this connection is expressed as

$$\omega = \frac{1}{|x|^2 + |y|^2}(\gamma(dX, X) + \gamma(dY, Y)), \tag{2.17}$$

which satisfies, as a connection form,

$$\omega(R_\xi(q)) = \xi, \tag{2.18}$$

$$\Phi_g^* w = \mathrm{Ad}_g \omega . \tag{2.19}$$

It is to be noted here that the structure group $SU(2)$ acts on $\dot{\mathbb{H}}^2$ to the left.

3. THE REDUCTION OF A PHASE SPACE

The phase space to start with is the cotangent bundle $T^*\dot{\mathbb{H}} \cong \dot{\mathbb{H}}^2 \times \mathbb{H}^2$, on which the conformal Kepler problem will be defined. For the cotangent vector $p = (p_X, p_Y) \in T_q^*(\dot{\mathbb{H}}^2)$ at $q = (X, Y) \in \dot{\mathbb{H}}^2$, the canonical one-form on $T^*\dot{\mathbb{H}}^2$ is expressed as

$$\theta = (p_X|dX) + (p_Y|dY) . \tag{3.1}$$

which will be abbreviated to $\theta = p \cdot dq$, where the center dot denotes the pairing. The $SU(2)$ action on $\dot{\mathbb{H}}^2$ lifts naturally to a symplectic action on $T^*\dot{\mathbb{H}}^2$;

$$\tilde{\Phi}_g(q,p) = (\Phi_g(q), \Phi_g(p)), \tag{3.2}$$

$$\tilde{\Phi}_g^*\theta = \theta. \tag{3.3}$$

For $\xi \in su(2)$, we denote the infinitesimal generator of $\tilde{\Phi}_{g(t)}$ with $g(t) = \exp t\xi$ by $\tilde{R}_\xi(q,p)$. Then, by using (2.6), the function $\theta(\tilde{R}_\xi(q,p))$ is written out as

$$\theta(\tilde{R}_\xi(q,p)) = \langle \gamma(p_X, X) + \gamma(p_Y, Y) | \xi \rangle. \tag{3.4}$$

On identifying $su(2)^*$ with $su(2)$ by the inner product (2.5), we then find the moment map, $J : T^*\dot{\mathbb{H}}^2 \to su(2)$ as

$$J(q,p) = \gamma(p_X, X) + \gamma(p_Y, Y). \tag{3.5}$$

As an immediate consequence of (2.4b) and (3.5), the J is Ad-equivariant;

$$J \circ \tilde{\Phi}_g = \mathrm{Ad}_g J. \tag{3.6}$$

Now that the setting is completed, we are to proceed to the reduction of $T^*\dot{\mathbb{H}}^2$ by the $SU(2)$ action (3.2). According to the reduction recipe [11], we are to take the submanifold $J^{-1}(\mu)$ for $\mu \in su(2)$, and to form the quotient manifold $J^{-1}(\mu)/G_\mu$ by the isotropy subgroup

$$G_\mu := \{g \in SU(2); \mathrm{Ad}_g\mu = \mu\}. \tag{3.7}$$

According as $\mu \neq 0$ or $\mu = 0$, G_μ is isomorphic to $U(1)$ or $SU(2)$.

To carry out the reduction program, we start by defining

$$\begin{aligned} N^\sharp &:= J^{-1}(0) \\ &= \{(q,p_0) \in T^*N \text{ with } N = \dot{\mathbb{H}}^2; p_0 \cdot R_\xi(q) = 0, \xi \in su(2)\}, \end{aligned} \tag{3.8}$$

which can be identified with the pull-back of the bundle $\pi : N := \dot{\mathbb{H}}^2 \to M := \dot{\mathbb{R}}^5$ to T^*M;

$$\begin{array}{ccccc} N^\sharp & \xrightarrow{\tau_N^\sharp} & N & = \dot{\mathbb{H}}^2 \\ \downarrow & & \downarrow & \\ T^*M & \xrightarrow{\tau_M} & M & = \dot{\mathbb{R}}^5 \end{array} \tag{3.9}$$

where τ_M is the natural projection, and $\tau_N^\sharp$ denotes the restriction of the natural projection $\tau_N : T^*N \to N$ to the subspace $N^\sharp$. Following Montgomery [7], one defines an isomorphism

$$N^\sharp \times su(2)^* \to T^*N$$

of vector bundles over N by

$$((q, p_0), \mu) \mapsto (q, p_0 + \omega_q^* \mu) , \tag{3.10}$$

where the connection form ω_q at q is considered as a linear map $T_q(\dot{\mathbf{H}}^2) \to su(2)$ and ω_q^* as its dual map. From this, the momentum manifold $J^{-1}(\mu)$ is diffeomorphic to $N^\sharp$,

$$J^{-1}(\mu) \cong N^\sharp \times \{\mu\} \cong N^\sharp , \tag{3.11}$$

so that for $\mu \neq 0$ the reduced phase space is given by

$$\begin{aligned} P_\mu := J^{-1}(\mu)/G_\mu &\cong N^\sharp/G_\mu \\ &\cong N^\sharp \times_{SU(2)} (SU(2)/U(1)) . \end{aligned} \tag{3.12}$$

In the last isomorphism, we have used the following lemma.

LEMMA 3.1. *Let $P \to M$ be a principal G-bundle and G_0 a closes subgroup. Then*

$$P/G_0 \cong P \times_G G/G_0 . \tag{3.13}$$

■

For $\mu = 0$, the reduced phase space becomes clearly $N^\sharp/SU(2)$. This and the isomorpshim (3.12) imply the following.

THEOREM 3.2. *The reduced phase space $P_\mu := J^{-1}(\mu)/G_\mu$ with $\mu \neq 0$ is diffeomorphic to the fibre bundle with base space $N^\sharp/SU(2) \cong T^*M = T^*\dot{\mathbf{R}}^5$ and fibre $SU(2)/U(1) \cong S^2$. For $\mu = 0$, the reduced phase space becomes $N^\sharp/SU(2) \cong T^*M$. The reduced symplectic form σ_μ will be given by (4.3) below.*

It is to be noted that the fibre S^2 of $P_\mu \to T^*M$ representes the internal degrees of freedom for the reduced system. We refer to this internal degrees of freedom as the isospin.

4. The $SU(2)$ KEPLER PROBLEM

As was anticipated in Introduction, the $SU(2)$ Kepler problem is reduced from the conformal Kepler problem on $T^*\dot{\mathbf{R}}^8$ by using the $SU(2)$ action $\tilde{\Phi}$. We start with the reduction of the symplectic form $d\theta$ and the kinetic energy on $T^*\dot{\mathbf{R}}^8$, and then define the $SU(2)$ Kepler problem.

Let $i_\mu : J^{-1}(\mu) \to T^*\dot{\mathbf{H}}^2$ be the inclusion map for a fixed $\mu \in su(2)$. Then from (3.10) the canonical one-form θ is pulled back to

$$\begin{aligned} i_\mu^*\theta &= (p_0 + \omega_q^*\mu) \cdot dq \\ &= p_0 \cdot dq + \langle \mu | \omega^\sharp \rangle , \end{aligned} \tag{4.1}$$

where $\omega^\sharp$ is the pull-back of the connection form ω on $\dot{\mathbf{H}}^2 = N$ through $\tau_N^\sharp$ (see (3.9)),

$$\omega^\sharp := (\tau_N^\sharp)^*\omega . \tag{4.2}$$

The reduced symplectic form σ_μ on $P_\mu = J^{-1}(\mu)/G_\mu$ is now determined by

$$dp_0 \wedge dq + \langle \mu | d\omega^\sharp \rangle = \pi_\mu^*\sigma_\mu , \tag{4.3}$$

where $\pi_\mu : J^{-1}(\mu) \to J^{-1}(\mu)/G_\mu$ is the natural projection. The first term $dp_0 \wedge dq$ in the left-hand side, however, can be viewed as a two-form on the base space T^*M. This is because the form $p_0 \cdot dq$ is horizontal one-form and invariant under the $SU(2)$ action $\tilde{\Phi}$, so that it can project to a one-form on the base space $T^*M \cong N^\sharp/SU(2)$. The second term $\langle \mu | d\omega^\sharp \rangle$ indeed projects to a two-form on P_μ. This can be verified as follows: First, we note that $\langle \mu | d\omega^\sharp \rangle$ is G_μ-inviariant;

$$\langle \mu | \tilde{\Phi}_g^* d\omega^\sharp \rangle = \langle \mu | d\omega^\sharp \rangle \quad \text{for } g \in G_\mu . \tag{4.4}$$

Second, it is horizontal with respect to $G_\mu = U(1)$, viz., for $\xi \in u(1)$ one has

$$\langle \mu | i(\tilde{R}_\xi(q, p)) d\omega^\sharp \rangle = 0 , \tag{4.5}$$

where $i()$ denotes the interior product. Both of (4.4) and (4.5) are consequences of

$$\tilde{\Phi}_g^* \omega^\sharp = \mathrm{Ad}_g \omega^\sharp \quad \text{for } g \in SU(2) , \tag{4.6}$$

and $\mathrm{Ad}_g \mu = \mu$ for $g \in G_\mu$. Equations (4.4) and (4.5) now imply that $\langle \mu | d\omega^\sharp \rangle$ projects to a two-form Σ_μ on P_μ;

$$\langle \mu | d\omega^\sharp \rangle = \pi_\mu^* \Sigma_\mu . \tag{4.7}$$

It is the two-form Σ_μ that is characteristic of the reduced symplectic form σ_μ. In fact, σ_μ depends on the isospin. To have a closer look at this, we have only to introduce the curvature form $\Omega = d\omega - \omega \wedge \omega$ to express $d\omega^\sharp$ as $d\omega^\sharp = \Omega^\sharp + \omega^\sharp$, where $\Omega^\sharp := (\tau_N^\sharp)^*\Omega$. Then, the term $\langle\mu|\omega^\sharp \wedge \omega^\sharp\rangle$ projects to the two-form the restriction of which to a fibre S^2 of $P_\mu \to T^*M$ turns out to be the canonical symplectic form on the (co)adjoint orbit through $\mu \in su(2)$ for $SU(2)$. In addition, note that the term $\langle\mu|\Omega^\sharp\rangle$ projects to a two-form horizontal to a fibre of $P_\mu \to T^*M$. Note also that, pulled back on $\dot{\mathbb{R}}^5$ through local cross sections in $\dot{\mathbb{H}}^2$, the Ω are known to provide Yang's monopole field [9]. Since Ω becomes the $BPST$ instanton, when restricted on S^4, we refer to Ω as $BPST$-Yang's monopole field.

We now proceed to the reduction of the kinetic energy on $T^*\dot{\mathbb{H}}^2$. Let K_q^* denote the inner product on $T_q^*(\dot{\mathbb{H}}^2)$, dual to K_q on $T_q(\dot{\mathbb{H}}^2)$. Denoting by $K^\sharp$ the isomorphism of $T_q^*(\dot{\mathbb{H}}^2)$ to $T_q(\dot{\mathbb{H}}^2)$, one has, for $p \in T_q^*(\dot{\mathbb{H}}^2)$,

$$K_q^*(p,p) = K_q(K_q^\sharp(p), K_q^\sharp(p)),$$

twice the kinetic energy on $T^*\dot{\mathbb{H}}^2$. Restricted on $J^{-1}(\mu)$, the kinetic energy turns out, after calculation, to be

$$K_q^*(p_0 + \omega_q^*\mu, p_0 + \omega_q^*\mu) = K_q^*(p_0, p_0) + \frac{1}{r}\langle\mu|\mu\rangle. \tag{4.8}$$

Note that the first term in the right-hand side can project to a function on the base space T^*M, since it is horizontal and invariant under the $SU(2)$ action. The second term is, of course, considered as a function on T^*M.

We are now in a position to define the $SU(2)$ Kepler problem. Let H_c be the Hamiltonian of the conformal Kepler problem on $T^*\dot{\mathbb{H}}^2$, which is given, for $(q,p) \in T^*\dot{\mathbb{H}}^2$, by

$$H_c = \frac{1}{2}\frac{1}{4r}K_q^*(p,p) - \frac{k}{r}, \tag{4.9}$$

where $r = |x|^2 + |y|^2$, and k is a positive constant. It is to be noted here that the conformal metric $4rK$ is used to define the kinetic energy and the distance. Then the reduced Hamiltonian H_μ on P_μ is determined by $i_\mu^* H_c = \pi_\mu^* H_\mu$, viz.,

$$\frac{1}{2}\frac{1}{4r}K_q^*(p_0, p_0) + \frac{1}{8r^2}\langle\mu|\mu\rangle - \frac{k}{r} = \pi_\mu^* H_\mu. \tag{4.10}$$

DEFINITION 4.1. *The $SU(2)$ Kepler problem is defined as a triple $(P_\mu, \sigma_\mu, H_\mu)$, where P_μ with $\mu \neq 0$ denotes the fibre bundle given in (3.12) and σ_μ and H_μ are* *determined by (4.3) and (4.10), respectively. If* $\mu = 0$ *, the* $SU(2)$ *Kepler problem becomes the standard Kepler problem defined on* $T^*\dot{\mathbf{R}}^5$.

To show why this system is referred to as the $SU(2)$ Kepler problem, we are to express H_μ in terms of the coordinates $(w_j), j = 0, \dots 4$. From (2.13) and (2.14), we observe that

$$\text{(4.11a)} \qquad K^\sharp(\pi^* dw_k) = 2H_k ,$$

$$\text{(4.11b)} \qquad K^*(\pi^* dw_j, \pi^* dw_k) = 4r\delta_{jk}, \quad j, k = 0, \dots, 4 .$$

Equation (4.11a) implies that $\pi^* dw_k, k = 0, \dots, 4$, span the horizontal subspace of $T_q^*(\dot{\mathbf{H}}^2)$ at every point $q \in \dot{\mathbf{H}}^2$ so that the condition of (3.8) implies that p_0 is put in the form

$$\text{(4.12a)} \qquad p_0 = \sum_{k=0}^{4} a_k \pi^* dw_k .$$

Equation (4.11b) then determines the coefficients a_k ;

$$\text{(4.12b)} \qquad a_k = \frac{1}{4r} K_q^*(p_0, \pi^* dw_k) .$$

We wish to show that a_k's project to the canonical momentum variables on T^*M . With the conformal metric $4rK$ on $\dot{\mathbf{H}}^2$, one can associate a metric B on $M = \dot{\mathbf{R}}^5$ by

$$\text{(4.13)} \qquad B_{\pi(q)}\left(\frac{\partial}{\partial w_j}, \frac{\partial}{\partial w_k}\right) = 4rK_q\left(\left(\frac{\partial}{\partial w_j}\right)^*, \left(\frac{\partial}{\partial w_k}\right)^*\right) ,$$

where $(\partial/\partial w_j)^*$ are the horizontal lifts given by (2.15). In the dual manner, one has

$$\text{(4.14)} \qquad B^*_{\pi(q)}(dw_j, dw_k) = \frac{1}{4r} K_q^*(\pi^* dw_j, \pi^* dw_k) .$$

Equations (4.11b) and (4.14) are put together to give

$$\text{(4.15)} \qquad B^*_{\pi(q)}(dw_j, dw_k) = \delta_{ik}, \qquad j, k = 0, \dots, 4 .$$

This shows that B is equal to the standard flat metric on $\dot{\mathbf{R}}^5$. Further, from (4.12) and (4.14), it follows that

$$\text{(4.16)} \qquad \frac{1}{4r} K_q^*(p_0, \pi^* dw_k) = B^*_{\pi(q)}(\bar{p}, dw_k) ,$$

where $\bar{p} \in T^*_{\pi(q)}M$ with $\pi^*\bar{p} = p_0$. From (4.12) and (4.16), a_k's are taken as canonical momentum variables on $T^*\dot{\mathbf{R}}^5$. We then set

$$a_k = \bar{p}_k := B^*_{\pi(q)}(\bar{p}, dw_k), \qquad k = 0, \ldots, 4 . \tag{4.17}$$

Now the Hamiltonian H_μ determined by (4.10) is put in the form

$$H_\mu = \frac{1}{2}\sum_{j=0}^{4} \bar{p}_j^2 + \frac{1}{8r^2}\langle\mu|\mu\rangle - \frac{k}{r} , \tag{4.18}$$

where $r = \|w\|$ (see (2.16)). This means that H_μ is equal to the Kepler Hamiltonian plus a centrifugal pontential $\langle\mu|\mu\rangle/8r^2$. For these reasons, we have referred to the reduced system $(P_\mu, \sigma_\mu, H_\mu)$ as the $SU(2)$ Kepler problem. In conclusion, we observe that σ_μ is expressed as

$$\sigma_\mu = \sum_{j=0}^{4} d\bar{p}_j \wedge dw_j + \Sigma_\mu . \tag{4.19}$$

5. NEGATIVE-ENERGY MANIFOLDS

In this section, we will study negative-energy manifolds for the $SU(2)$ Kepler problem. Since the $SU(2)$ Kepler problem is reduced from the conformal Kepler problem, we start with the study of negative-energy manifolds for the conformal Kepler problem. Let H be the harmonic oscillator Hamiltonian;

$$H = \frac{1}{2}K^*_q(p, p) + \frac{\lambda^2}{2} r , \tag{5.1}$$

where λ is a positive constant, and $r = |x|^2 + |y|^2$. Then the Hamiltonians H and H_c are related by

$$4r\left(H_c + \frac{\lambda^2}{8}\right) = H - 4k , \tag{5.2}$$

which implies that, in $\dot{\mathbf{H}}^2 \times \mathbf{H}^2$, the negative-energy manifold $H_c^{-1}(-\lambda^2/8)$ for the conformal Kepler problem coincides with the energy manifold $H^{-1}(4k)$ for the harmonic oscillator. Thus, if $H^{-1}(4k)$ does not intersect the set $\{0\} \times \mathbf{H}^2$, the negative-energy manifold $H_\mu^{-1}(-\lambda^2/8)$ for the $SU(2)$ Kepler problem can be obtained as the quotient space $J^{-1}(\mu) \cap H^{-1}(4k)/G_\mu$ because of $i^*_\mu H_c = \pi^*_\mu H_\mu$.

We call $J^{-1}(\mu)\cap H^{-1}(4k)$ the energy-momentum manifold. From (4.8) and (5.1), it is determined by

$$\frac{1}{2}K_q^*(p_0,p_0)+\frac{1}{2r}|\mu|^2+\frac{\lambda^2}{2}r=4k\,,$$

where $|\mu|^2=\langle\mu|\mu\rangle$. From this, one has

(5.3) $$rK_q^*(p_0,p_0)+\lambda^2\left(r-\frac{4k}{\lambda^2}\right)^2=\left(\frac{4k}{\lambda}\right)^2-|\mu|^2\,,$$

so that

(5.4) $$4k\geq\lambda|\mu|\,,$$

where the equality holds if and only if

(5.5) $$p_0=0\qquad r=4k/\lambda^2\,.$$

Further, Eq. (5.3) implies that if $\mu\neq 0$, then $r\neq 0$, viz., no collision orbits exist, so that $H_c^{-1}(-\lambda^2/8)=H^{-1}(4k)$ exactly. However, if $\mu=0$, collision orbits can occur. In this case, one has $H_c^{-1}(-\lambda^2/8)=H^{-1}(4k)$ out of $\{0\}\times\mathbb{H}^2$. We then understand that the energy-momentum manifold for the conformal Kepler problem, $J^{-1}(0)\cap H_c^{-1}(-\lambda^2/8)$, should be regularized to be $J^{-1}(0)\cap H^{-1}(4k)$ in the whole space $\mathbb{H}^2\times\mathbb{H}^2$. In what follows, we take the energy-momentum manifold as regularized in $\mathbb{H}^2\times\mathbb{H}^2$ if necessary.

We are to study the energy-momentum manifold under the condition (5.4). The case of $4k>\lambda|\mu|$ is dealt with first, and the case of $4k=\lambda|\mu|$ is considered after. Let us introduce the variables

(5.6) $$\zeta_X=p_X+i\lambda X,\qquad \zeta_Y=p_Y+i\lambda Y\,,$$

which are variables in $\mathbb{H}+i\mathbb{H}\cong M(2,\mathbb{C})$, the linear space of 2×2 complex matrices. Then, $\mathbb{H}^2\times\mathbb{H}^2$ is taken to be $M(2,\mathbb{C})\times M(2,\mathbb{C})$. Using ζ_X and ζ_Y, we find an easy but important equation

(5.7) $$\frac{1}{2}(\zeta_X\zeta_X^*+\zeta_Y\zeta_Y^*)=HI-i\lambda J\,,$$

where I is the 2×2 identity matrix. Hence, the energy-momentum manifold $J^{-1}(\mu)\cap H^{-1}(4k)$ is determined by

(5.8) $$\zeta_X\zeta_X^*+\zeta_Y\zeta_Y^*=2(4kI-i\lambda\mu)\,.$$

It is an easy matter to show that the left-hand side is invariant under $U(4)$ action on $M(2,\mathbb{C})\times M(2,\mathbb{C})$ to the right, where the right $U(4)$ action is given by

$$(\zeta_X,\zeta_Y)\mapsto(\zeta_X,\zeta_Y)\begin{pmatrix}A & B\\ C & D\end{pmatrix} \tag{5.9}$$

with A,B,C, and D 2×2 complex matrices subject to the condition of $U(4)$ altogether.

We show the following.

PROPOSITION 5.1. *On the energy-momentum manifold $J^{-1}(\mu)\cap H^{-1}(4k)$ with $4k>\lambda|\mu|$, $SU(4)$ acts transitevely to the right.*

Proof. We note first that under the condition $4k>\lambda|\mu|$, the matrix $4kI-i\lambda\mu$ is a positive-definite Hermitian matrix, so that the square root Hermitian matrix $[4kI-i\lambda\mu]^{1/2}$ exists. Let (ζ_X,ζ_Y) be an arbitrary point subject to (5.8). Then using the square root matrix, we obtain

$$\begin{aligned}&[2(4kI-i\lambda\mu)]^{-1/2}\zeta_X\zeta_X^*[2(4kI-i\lambda\mu)]^{-1/2}\\&+[2(4kI-i\lambda\mu)]^{-1/2}\zeta_Y\zeta_Y^*[2(4kI-i\lambda\mu)]^{-1/2}=I.\end{aligned} \tag{5.10}$$

Hence, the matrix

$$g:=\begin{pmatrix}[2(4kI-i\lambda\mu)]^{-1/2}\zeta_X & [2(4kI-i\lambda\mu)]^{-1/2}\zeta_y\\ \eta_1 & \eta_2\end{pmatrix} \tag{5.11}$$

can be made into a 4×4 unitary unimodular matrix, if 2×2 matrices η_1 and η_2 are suitably chosen. Thus, for g given by (5.11), one has

$$([2(4kI-i\lambda\mu)]^{1/2},0)g=(\zeta_X,\zeta_Y), \tag{5.12}$$

which proves the transitivity of the $SU(4)$ right action. This ends the proof. ∎

On Prop. 5.1, we are allowed to treat the energy-momentum manifold as a homogeneous space with $SU(4)$. For convenience' sake, we take the energy-momentum manifold to be $J^{-1}(0)\cap H^{-1}(4k)$, since $J^{-1}(\mu)$ and $J^{-1}(0)=N^\sharp$ are diffeomorphic because of (3.11). With this in mind, we will denote the energy-momentum manifold by $N_k^\sharp$. In the below, we choose to use the notation $N_k^\sharp$ in the case of $4k<\lambda|\mu|$. Take a point $(\sqrt{8k}I,0)$ of $N_k^\sharp$. Then the isotropy subgroup of $SU(4)$ at this point turns out to be isomorphic to $SU(2)$. Thus we have the following.

THEOREM 5.2. *The energy-momentum manifold* $J^{-1}(\mu) \cap H^{-1}(4k)$ *with* $4k > \lambda|\mu|$ *is diffeomorphic to the homogeneous space* $SU(4)/SU(2)$ *, and hence to the complex Stiefel manifold,* $V_2(\mathbb{C}^4)$ *, of orthonormal two-frames in* $\mathbb{C}^4$ *;*

(5.13) $$J^{-1}(\mu) \cap H^{-1}(4k) \cong SU(4)/SU(2) \cong V_2(\mathbb{C}^4) .$$

■

Now that the energy-momentum manifold $N_k^\sharp$ is found out, we proceed to the quotient space $N_k^\sharp/G_\mu$ to get the negative-energy manifold $H_\mu^{-1}(-\lambda^2/8)$ for the $SU(2)$ Kepler problem. However, we have to remark that the G_μ is a subgroup of the structure group $SU(2)$, which acts on $\mathbb{H}^2 \times \mathbb{H}^2 \cong M(2, \mathbb{C})$ to the left. In fact, from (3.2) and (5.6), one has the $SU(2)$ action, for $g \in SU(2)$, in the manner

(5.14) $$(\zeta_X, \zeta_Y) \mapsto (g\zeta_X, g\zeta_Y) .$$

Let us here be reminded that the $SU(4)$ action (5.9) is to the right.

In what follows, we therefore have to deal with the $SU(2)$ left action and the $SU(4)$ right action simultaneously. To get acquainted with the two-sided action, we take up $N_k^\sharp$ again, on which, clearly, $SU(2) \times SU(4)$ acts transitively. Then, the isotropy subgroup of $SU(2) \times SU(4)$ at a poit $(\sqrt{8k}I, 0)$ of $N_k^\sharp$ is shown to be

(5.15) $$\left\{ \left(g, \begin{pmatrix} g^{-1} & 0 \\ 0 & h \end{pmatrix} \right) ; g, h \in SU(2) \right\} \subset SU(2) \times SU(4) ,$$

which is isomorphic to $SU(2) \times SU(2)$. Hence, one has

$$N_k^\sharp \cong SU(2) \times SU(4)/SU(2) \times SU(2) \cong SU(4)/SU(2) ,$$

the same result as (5.13). Now we are ready to treat the quotient space $N_k^\sharp/G_\mu$ as a homogeneous space. We start with the case of $G_\mu = U(1)$. Let $U(1) \times SU(4)$ act transitevely on $N_k^\sharp$, and therefore on $N_k^\sharp/U(1)$. Take a point $p_0 = (\sqrt{8k}I, 0)$ of $N_k^\sharp$. We denote by $[p_0]$ the equivalence class of p_0 , a point of $N_k^\sharp/U(1)$. In a similar method to (5.15), the isotropy subgroup of $U(1) \times SU(4)$ at $[p_0]$ turns out to be isomorphic to $U(1) \times SU(2)$, so that

$$\begin{aligned} N_k^\sharp/U(1) &\cong U(1) \times SU(4)/U(1) \times U(1) \times SU(2) \\ &\cong SU(4)/U(1) \times SU(2) . \end{aligned}$$

Thus we obtain the following.

THEOREM 5.3. *The negative-energy manifold* $H_\mu^{-1}(-\lambda^2/8)$ *with* $4k > \lambda|\mu|$ *and* $\mu \neq 0$ *for the* $SU(2)$ *Kepler problem is a homogeneous space with* $SU(4)$ *;*

$$H^{-1}(-\lambda^2/8) \cong SU(4)/U(1) \times SU(2)\ . \tag{5.16}$$

■

In case of $\mu = 0$, G_μ is equal to $SU(2)$, so that the quotient space $N_k^\sharp/SU(2)$ gives the «regularized» negative-energy manifold $\bar{H}_0^{-1}(-\lambda^2/8)$, where the overbar denotes the regularization, which we have here to mention. As was stated in the paragraph after Eq. (5.5), regularization has been made for the energy-momentum manifold $N_k^\sharp$ in the case of $\mu = 0$, so that the $N_k^\sharp/SU(2)$ should be taken also as regularizd. We continue to discuss $N_k^\sharp/SU(2)$ as a homogeneous space. Let $SU(2) \times SU(4)$ act on $N_k^\sharp$ in the two-sided manner, and therefore on the quotient space $N_k^\sharp/SU(2)$. Then, in the same manner as in the case of $\mu \neq 0$, it tunrs out that

$$\begin{aligned} N_k^\sharp/SU(2) &\cong SU(2) \times SU(4)/SU(2) \times SU(2) \times SU(2) \\ &\cong SU(4)/SU(2) \times SU(2) \\ &\cong \mathrm{Spin}(6)/\mathrm{Spin}(4) \\ &\cong SO(6)/SO(4) \cong V_2(\mathbf{R}^6)\ , \end{aligned} \tag{5.17}$$

where $V_2(\mathbf{R}^6)$ is the real Stiefel manifold of orthonormal two-frames in $\mathbf{R}^6$. Thus we have the following.

PROPOSITION 5.4. *The regularized negative-energy manifold* $\bar{H}_0^{-1}(-\lambda^2/8)$, *the overbar denoting the regularization, for the* $SU(2)$ *Kepler problem with* $\mu = 0$ *(i.e., the usual Kepler problem) is diffeomorphic to a homogeneous manifold;*

$$\bar{H}_0^{-1}(-\lambda^2/8) \cong SU(4)/SU(2) \times SU(2) \cong V_2(\mathbf{R}^6)\ . \tag{5.18}$$

■

We notice here that, for the usual Kepler problem in n dimensions, the regularized negative-energy manifold is known to be diffeomorphic to the unit sphere bundle over S^n [13], and hence to the Stiefel manifold $V_2(\mathbf{R}^{n+1})$ of orthonormal two-frames in $\mathbf{R}^{n+1}$. Since, in the case of $\mu = 0$, the $SU(2)$ Kepler problem becomes the usual Kepler problem on $T^*\dot{\mathbf{R}}^5$, Prop. 5.4 is a special case of the known result.

So far, we have considered two quotient spaces $N_k^\sharp/U(1)$ and $N_k^\sharp/SU(2)$. Now, in an analogous fashion to (3.13), we relate these two space as follows:

$$N_k^\sharp/U(1) \cong N_k^\sharp \times_{SU(2)} (SU(2)/U(1))\ , \tag{5.19}$$

which imples that $N_k^\sharp/U(1)$ has a fibre bundle structure with base space $N_k^\sharp/SU(2) \cong V_2(\mathbf{R}^6)$ and fibre $SU(2)/U(1) \cong S^2$. Therefore, Th. 5.3 and Prop. 5.4 are put together to imply the following.

PROPOSITION 5.5. *The negative-energy manifold $H_\mu^{-1}(-\lambda^2/8)$ with $4k > \lambda|\mu|$ and $\mu \neq 0$ has a fibre bundle structure with base space $V_2(\mathbf{R}^6)$ and fibre S^2 .* ∎

In the rest of this section, we treat the case of $4k = \lambda|\mu|$. As was shown in (5.5), this case occurs if and only if $p_0 = 0$ and $r = |x|^2 + |y|^2 = 4k/\lambda^2$. Hence, from (3.10), we obtain the section in $T^*\dot{\mathbf{H}}^2$,

$$\{(q, \omega_q^*\mu); q = (X,Y), (X|X) + (Y|Y) = 4k/\lambda^2\} , \tag{5.20}$$

which is diffeomorphic to the round sphere $|x|^2 + |y|^2 = 4k/\lambda^2$ in $\mathbf{H}^2 \cong \mathbf{R}^8$. Thus we have the follwoing.

PROPOSITION 5.6. *The energy-momentum manifold $J^{-1}(\mu) \cap H^{-1}(4k)$ with $4k = \lambda|\mu|$ is diffeomorphic to the sphere S^7 .*

∎

The condition $4k = \lambda|\mu|$ means that $\mu \neq 0$, so that $G_\mu = U(1)$. Hence the negative-energy manifold $H_\mu^{-1}(-\lambda^2/8)$ is obtained as the quotiens space $J^{-1}(\mu) \cap H^{-1}(4k)/U(1)$, which is equal to $S^7/U(1) \cong \mathbf{C}P^3$, the complex projective space, being known to be described as a homogeneous space with $SU(4)$ (see also Th. 7.3 in Sec. 7). Further, from (4.12 a) and (4.17), the condition $p_0 = 0$ implies that $\bar{p}_k = 0$, $k = 0, \ldots, 4$, so that the $\mathbf{C}P^3$ represents the set of equilibrium states for the $SU(2)$ Kepler problem. Thus we have the following.

THEOREM 5.7. *If $4k = \lambda|\mu|$, the negative-energy manifold $H_\mu^{-1}(-\lambda^2/8)$ becomes the complex projective space $\mathbf{C}P^3$, which represents the set of equilibrium states.* ∎

6. ISOENERGETIC ORBIT SPACES

We return to Eq. (5.2). From this the Hamiltonian vector fields X_H and X_{H_c} are related, on the energy manifold $H^{-1}(4k) = H_c^{-1}(-\lambda^2/8)$, by

$$X_H = 4rX_{H_c} . \tag{6.1}$$

This implies that the folw of X_{H_c} coincides with that of X_H within a change of parameters. Thus we are allowed to treat the flow of X_H instead of that of X_{H_c} . As is shown easily, the harmonic oscillator flows provide the $U(1)$ action on $M(2,\mathbb{C}) \times M(2,\mathbb{C})$,

$$(\zeta_X,\zeta_Y) \mapsto (e^{i\lambda t}\zeta_X, e^{i\lambda t}\zeta_Y) . \tag{6.2}$$

This action, of course, induces a $U(1)$ action on the energy-momentum manifold, since J and H are invariant under (6.2). Moreover, the action (6.2) is commutative with the $SU(2)$ action (5.14), so that the induced $U(1)$ action on the energy-moment um manifold projects to an action on the quotient space $N_k^\sharp/G_\mu$, viz., on the negative-energy manifold $H^{-1}(-\lambda^2/8)$. Thus we conclude the follwoing, in the case of $4k > \lambda|\mu|$.

PROPOSITION 6.1. *All the orbits of the $SU(2)$ Kepler problem of negative energy with $4k > \lambda|\mu|$ are closed, and provides a $U(1)$ action on the negative-energy manifold $N_k^\sharp/U(1)$ or $N_k^\sharp/SU(2)$ according as $\mu \neq 0$ or $\mu = 0$.* ∎

The isoenergetic orbit spaces of the $SU(2)$ Kepler problem with $4k > \lambda|\mu|$ are then given by $N_k^\sharp/U(1) \times U(1)$ or $N_k^\sharp/SU(2) \times U(1)$, according as $\mu \neq 0$ or $\mu = 0$.

We wish to study these orbit spaces as homogeneous spaces. Let us pick up $N_k^\sharp/U(1) \times U(1)$ first. The two-sided transitive action of $U(1)\times U(1)\times SU(4)$ on $N_k^\sharp$ induces an action on the quotient space $N_k^\sharp/U(1) \times U(1)$. Take a point $p_0 = (\sqrt{8k}I, 0)$ of $N_k^\sharp$ and denote by $[p_0]$ the projected point on $N_k^\sharp/U(1) \times U(1)$. Then the isotropy subgroup of $U(1) \times U(1) \times SU(4)$ is found to be isomorphic to $U(1) \times U(1) \times S(U(1)\times U(1)\times U(2))$, where $S(U(1)\times U(1)\times U(2))$ is the subgroup of $SU(4)$, which is expressed as

$$\left\{ \begin{pmatrix} e^{i\varphi}g(t) & 0 \\ 0 & h \end{pmatrix} ; g(t) \in U(1) \subset SU(2),\ h \in U(2) \right\} \cap SU(4) .$$

Then it follows that

$$\begin{aligned} N_k^\sharp/U(1) \times U(1) &\cong U(1) \times U(1) \times SU(4)/U(1)\times \\ &\qquad U(1) \times S(U(1) \times U(1) \times U(2)) \\ &\cong SU(4)/S(U(1) \times U(1) \times U(2)) \\ &\cong F_{1,2}(\mathbb{C}^4) , \end{aligned} \tag{6.3}$$

where $F_{1,2}(\mathbb{C}^4)$ denotes the complex flag manifold of the pairs (U_1, U_2) with U_1 a one-dimensional linear subspace of $\mathbb{C}^4$ and U_2 a two-dimensional linear subspace containing U_1. Thus we have the following.

THEOREM 6.2. *The isoenergetic orbit space for the $SU(2)$ Kepler problem of negative energy with $4k > \lambda|\mu|$ and $\mu \neq 0$ is a homogeneous manifold $SU(4)/S(U(1) \times U(1) \times U(2))$, and diffeomorphic to the complex flag manifold, $F_{1,2}(\mathbb{C}^4)$, consisting of all the pairs (U_1, U_2) with U_1 a one-dimensional linear subspace of $\mathbb{C}^4$ and U_2 a two-dimensional linear subspace containing U_1.* ∎

For $\mu = 0$, the isoenergetic orbit space is given, in a similar manner to the above, as follows:

$$
\begin{aligned}
N_k^{\sharp}/SU(2) \times U(1) &\cong SU(2) \times U(1) \times SU(4)/SU(2) \times \\
&\qquad U(1) \times S(U(2) \times U(2)) \\
&\cong SU(4)/S(U(2) \times U(2)) \\
&\cong G_2(\mathbb{C}^4),
\end{aligned} \tag{6.4}
$$

where $G_2(\mathbb{C}^4)$ denotes the complex Grassmann manifold of complex two-planes in $\mathbb{C}^4$. The proof is easy to carry out. The isomorphisms (6.4) then result in the following.

PROPOSITION 6.3. *The isoenergetic orbit space for the $SU(2)$ Kepler problem of negative energy with $\mu = 0$ (i.e., the usual Kepler problem) is a homogeneous manifold $SU(4)/S(U(2) \times U(2))$, and diffeomorphic to the complex Grassmann manifold $G_2(\mathbb{C}^4)$, of complex two-planes in $\mathbb{C}^4$.* ∎

These two orbit spaces obtained above are related, on using Lemma 3.1, by

$$
\begin{aligned}
&N_k^{\sharp}/U(1) \times U(1) \cong \\
&N_k^{\sharp} \times_{SU(2)\times U(1)} (SU(2) \times U(1)/U(1) \times U(1)).
\end{aligned} \tag{6.5}
$$

This and (6.4) imply the following.

PROPOSITION 6.4. *The isoenergetic orbit space, $F_{1,2}(\mathbb{C}^4)$, for the $SU(2)$ Kepler problem of negative energy with $4k > \lambda|\mu|$ and $\mu \neq 0$ has a fibre bundle structure with base space $G_2(\mathbb{C}^4)$ and fibre S^2.* ∎

In conclusion, we consider the isoenergetic orbit space in the case of $4k = \lambda|\mu|$. As is known form Th. 5.7, the orbits of the $SU(2)$ Kepler problem are all equilibrium points, so that the orbit space coincides with the negative-energy manifold itself.

THEOREM 6.5. *In the case of $4k = \lambda|\mu|$, the isoenergetic orbit space for the $SU(2)$ Kepler problem is equal to the negative-energy manifold $\mathbb{C}P^3$.* ∎

7. COADJOINT STRUCTURE

In the last section we have obtained the isoenergetic orbit spaces for the $SU(2)$ Kepler problem of negative energy. From the reduction point of view, the isoenergetic orbit spaces, $N_k^\sharp/U(1)\times U(1)$ and $N_k^\sharp/SU(2)\times U(1)$, can be viewed as reduced phase spaces associated with the $U(1)\times SU(2)$ action, which is defined by (5.14) and (6.2). Equation (5.7), in fact, gives the moment map associated with the $U(1)\times SU(2)$ action. We show this observation below. Take the variables (ζ_X, ζ_Y) introduced in (5.6). Then the canonical symplectic form, $d\theta$, given through (3.1) is expressed as

$$(7.1)\qquad d\theta = \frac{i}{4\lambda}\mathrm{tr}(d\zeta_X \wedge d\zeta_X^* + d\zeta_Y \wedge d\zeta_Y^*) .$$

We set

$$(7.2)\qquad \Theta = \frac{i}{4\lambda}\mathrm{tr}(\zeta_X d\zeta_X^* + \zeta_Y d\zeta_Y^*) ,$$

which is cohomologous to θ; $d\theta = d\Theta$.

Set, for $g \in U(1)\times SU(2)$, a covering group of $U(2)$, and $\zeta = (\zeta_X, \zeta_Y)$,

$$(7.3)\qquad \tilde{\Phi}_g(\zeta) = (g\zeta_X, g\zeta_Y) .$$

Then one has $\tilde{\Phi}_g^*\Theta = \Theta$. For $\xi \in u(2) \cong u(1)\oplus su(2)$, we denote the infinitesimal generator of $\exp t\xi$ on $T := M(2,\mathbb{C})\times M(2,\mathbb{C})$ by $\xi_T(\zeta)$. Then the function $\Theta(\xi_T)$ are written out to provide

$$(7.4)\qquad \Theta(\xi_T) = \frac{1}{2\lambda}\langle \tilde{J}(\zeta)|\xi\rangle ,$$

where $\langle|\rangle$ is the inner product defined on $u(2)$ in the same manner as (2.5), and thereby $u(2)^*$ and $u(2)$ are identified, and further

$$(7.5)\qquad \tilde{J}(\zeta) = i(\zeta_X\zeta_X^* + \zeta_Y\zeta_Y^*) .$$

Thus one has the momentum map $\tilde{J} : T \to u(2)$. Hence Eq. (5.7) is put in the form

$$(7.6)\qquad \tilde{J}(\zeta) = 2(iHI + \lambda J) ,$$

so that the energy-momentum manifold $N_k^\sharp = J^{-1}(\mu)\cap H^{-1}(4k)$ is expressed as $\tilde{J}^{-1}(\tilde{\mu})$ with $\tilde{\mu} := 2(4ki + \lambda\mu)$. Moreover, it is clear that $\tilde{J}$ is adjoint-equivariant for the $U(1)\times SU(2)$ action;

$$\tilde{J}(g\zeta) = \mathrm{Ad}_g\tilde{J}(\zeta) \quad \text{for} \quad g \in U(1)\times SU(2) .$$

For $\tilde{\mu} = 2(4ik + \lambda\mu) \in u(1) \oplus su(2)$, the isotropy subgroup $G_{\tilde{\mu}}$, of $U(1) \times SU(2)$ acting on $u(2)$ proves to be isomorphic with $U(1) \times SU(2)$ or $U(1) \times U(1)$, according as $\mu = 0$ or $\mu \neq 0$. Hence, the reduced phase space $\tilde{J}^{-1}(\tilde{\mu})/G_{\tilde{\mu}}$ turns out to be identical with the isoenergetic orbit space, $N_k^\dagger/U(1) \times U(1)$ or $N_k^\dagger/U(1) \times SU(2)$. We wish to show in this section that those spaces are realized as coadjoint orbits of $U(4)$.

Now we turn to the $U(4)$ right action (5.9). For $\zeta = (\zeta_X, \zeta_Y)$ and $h \in U(4)$, we denote the right action by

$$D_h(\zeta) = \zeta h^{-1}. \tag{7.7}$$

Then, one easily observes that Θ is invariant under $D_h, h \in U(4)$; $D_h^*\Theta = \Theta$. For $h(t) = \exp(-t\eta)$, $\eta \in u(4)$, we denote by η_T, the infinitesimal generator of $D_{h(t)}$ on $T = M(2, \mathbb{C}) \times M(2, \mathbb{C})$. We now set

$$\eta = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where $\alpha, \beta, \gamma, \delta$ are 2×2 complex matrices subject to the condition of $u(4)$ altogether. Then the function $\Theta(\eta_T)$ turns out, after calculation along with (7.2), to be

$$\Theta(\eta_T) = \frac{1}{4\lambda}\mathrm{tr}\left[\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\left(i\begin{pmatrix} \zeta_X^*\zeta_X & \zeta_X^*\zeta_Y \\ \zeta_Y^*\zeta_X & \zeta_Y^*\zeta_Y \end{pmatrix}\right)^*\right]. \tag{7.8}$$

This provides us with the moment map associated with the $U(4)$ right action as follows: define the inner product on $u(4)$ by

$$\langle \xi | \eta \rangle = \frac{1}{4}\mathrm{tr}(\xi\eta^*), \quad \xi, \eta \in u(4), \tag{7.9}$$

and thereby identify $u(4)^*$ with $u(4)$. Then Eq. (7.8) is put in the form

$$\Theta(\eta_T) = \frac{1}{\lambda}\langle \eta | F(\zeta) \rangle,$$

where

$$F(\zeta) = i\begin{pmatrix} \zeta_X^*\zeta_X & \zeta_X^*\zeta_Y \\ \zeta_Y^*\zeta_X & \zeta_Y^*\zeta_Y \end{pmatrix} = i\begin{pmatrix} \zeta_X^* \\ \zeta_Y^* \end{pmatrix}(\zeta_X, \zeta_Y). \tag{7.10}$$

Thus we have found the moment map $F : T \to u(4)$ expressed as (7.10). The following is easy to prove.

LEMMA 7.1. *The moment maps, $\tilde{J}$ and F, associated with the $U(1) \times SU(2)$ left the $U(4)$ right actions are subject to*

$$\tilde{\Phi}_g^* \tilde{J} = \mathrm{Ad}_g \tilde{J}, \quad D_h^* \tilde{J} = \tilde{J}, \tag{7.11}$$

$$\tilde{\Phi}_g^* F = F, \quad D_h^* F = \mathrm{Ad}_h F, \tag{7.12}$$

for $g \in U(1) \times SU(2)$ and $h \in U(4)$. ■

Now that we have completed the setting-up, we proceed to describe the isoenergetic orbit spaces, obtained in Sec. 6, as coadjoint orbits of $U(4)$. We start with the energy-momentum manifold $J^{-1}(\mu) \cap H^{-1}(4k)$ with $4k > \lambda|\mu|$. From Prop. 5.1 along with (5.12), one has

$$J^{-1}(\mu) \cap H^{-1}(4k) = \{\zeta_0 h;\ h \in U(4)\}, \tag{7.13}$$

where

$$\zeta_0 = \left([2(4kI - i\lambda\mu)]^{1/2}, 0\right). \tag{7.14}$$

Then, on using (7.11) and (7.13), it turns out that $F(J^{-1}(\mu) \cap H^{-1}(4k))$ is a (co)adjoint orbit through $F(\zeta_0)$ in $\mu(4)^* \cong u(4)$. By the definition (7.10) of F together with (7.14), $F(\zeta_0)$ takes the form

$$F(\zeta_0) = 4i \begin{pmatrix} 4kI - i\lambda\mu & 0 \\ 0 & 0 \end{pmatrix}. \tag{7.15}$$

Incidentally, the eigenvalues of the Hermitian matrix $4kI - i\lambda\mu$ are $4k - \lambda|\mu|$ and $4k + \lambda|\mu|$, so that, after the diagonalization of $4kI - i\lambda\mu$, this (co)adjoint orbit can be described as

$$\{\mathrm{Ad}_h \eta_\mu;\ h \in U(4)\},$$

where

$$\eta_\mu = 4i \begin{pmatrix} 4k - \lambda|\mu| & 0 & 0 & 0 \\ 0 & 4k - \lambda|\mu| & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7.16}$$

According as $\mu = 0$ or $\mu \neq 0$, the isotropy subgroup of $U(4)$ at $\eta_\mu \in u(4)$ can be shown to be isomorphic to $U(2) \times U(2)$ or $U(1) \times U(1) \times U(2)$. Thus we have the following.

THEOREM 7.2. *The image of the energy-momentum manifold* $J^{-1}(\mu) \cap H^{-1}(4k)$ *with* $4k > \lambda|\mu|$ *under the moment map* F *associated with the* $U(4)$ *right action is a (co)adjoint orbit of* $U(4)$ *, whic is diffeomorphic to* $U(4)/U(2) \times U(2) \cong G_2(\mathbb{C}^4)$ *, the Grassmann manifold, or* $U(4)/U(1) \times U(1) \times U(2) \cong F_{1,2}(\mathbb{C}^4)$ *, the flag manifold, according as* $\mu = 0$ *or* $\mu \neq 0$ *. These are diffeomorphic to the isoenergetic orbit spaces given in Prop. 6.3 and Th. 6.2, respectively.* ■

In the case of $4k = \lambda|\mu|$, we have to go to an alternative method. Let

$$u_1 = x_0 + ix_1, \quad u_2 = x_2 + ix_3, \qquad v_1 = y_0 + iy_1, \quad v_2 = y_2 + iy_3, \tag{7.17}$$

and

$$u = (u_1, u_2), \quad v = (v_1, v_2), \tag{7.18}$$

row vectors. We take $\mathbb{C}^4$ to be the set of pairs (u, v). Then $U(4)$ acts on $\mathbb{C}^4$ to the right, naturally. We now define a $u(4)$-valued function by

$$F_0(u, v) = i \begin{pmatrix} u^* \\ v^* \end{pmatrix} (u, v) . \tag{7.19}$$

Then, for the right $U(4)$ action, F_0 is subject to the transformation

$$F_0\left((u, v)g^{-1}\right) = gF_0(u, v)g^{-1}, \quad g \in U(4) . \tag{7.20}$$

Now we return to the sphere stated in Prop. 5.6. The S^7 is realized as $|u|^2 + |v|^2 = 4k/\lambda^2$, where $|u|^2 = |u_1|^2 + |u_2|^2$, etc., on which $U(4)$ acts transitevely to the right. Fix a point p_0 of S^7 with $\mu = \left((4k/\lambda^2)^{1/2}, 0\right)$ and $v = 0$. Then

$$\xi_0 := F_0(p_0) = i \begin{pmatrix} 4k/\lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

Hence the image of S^7 under the map F_0 is put in the form

$$\{\mathrm{Ad}_g \xi_0;\ g \in U(4)\}, \tag{7.21}$$

a (co)adjoint orbit. The isotropy subgroup of $U(4)$ at ξ_0 is $U(1) \times U(3)$, and therefore the (co)adjoint orbit is diffeomorphic to $U(4)/U(1) \times U(3) \cong \mathbb{C}P^3$. Thus we realize the orbit space stated in Th. 6.3 as a (co)adjoint orbit of $U(4)$.

PROPOSITION 7.3. *The isoenergetic orbit space for the $SU(2)$ Kepler problem with $4k = \lambda|\mu|$ is also realized as a (co)adjoint orbits of $U(4)$.* ■

In conclusion, we show in the case of $4k > \lambda|\mu|$ that the Kirillov-Kostant-Souriau form κ on the (co)adjoint orbit of $U(4)$ is related with the standard symplectic form $d\theta$ on $\mathbb{R}^8$.

PROPOSITION 7.4. *Let $i_{k,\mu}$ be the inclusion map of the energy-momentum manifold $J^{-1}(\mu) \cap H^{-1}(4k)$ with $4k > \lambda|\mu|$ to the phase space $T = M(2, \mathbb{C}) \times M(2, \mathbb{C})$. Then the standard symplectic form $d\theta$ on T is related to the Kirillov-Kostant-Soriau form κ on respective (co)adjoint orbits in $u(4)$, through the moment map (7.10), by*

$$i_{k,\mu}^{*} d\theta = \frac{1}{\lambda} F^{*}\kappa . \tag{7.22}$$

Proof. Let η and η_T be a vector in $u(4)$ and the infinitesimal generator on T. Then under the mapping $F : T \to u(4)$, one has

$$F_{*}\eta_T = -\eta_{u(4)} , \tag{7.23}$$

where $\eta_{u(4)}$ is the infinitesimal generator of the adjoint action of $\exp t\eta$ on $u(4)$. The Kirillov-Kostant-Souriau form κ on the (co)adjoint orbit through ν is now defined for $\eta_{u(4)}$ and $\xi_{u(4)}$, by

$$\kappa_{\nu}\left(\eta_{u(4)}, \xi_{u(4)}\right) = \langle \nu | [\eta, \xi] \rangle . \tag{7.24}$$

We are ready to prove (7.22). Let η_T and ξ_T be infinitesimal generators, which are tangent vectors at each point ζ to the energy-momentum manifold $J^{-1}(\mu) \cap H^{-1}(4k)$. Then from (7.1) and (7.9) it follows that

$$d\theta_{\zeta}(\eta_T, \xi_T) = \frac{1}{\lambda} \langle F(\zeta) | [\eta, \xi] \rangle . \tag{7.25}$$

Thus one has, from (7.23)-(7.25),

$$d\theta_{\zeta}(\eta_T, \xi_T) = \frac{1}{\lambda} F^{*}\kappa(\eta_T, \xi_T) .$$

Since $U(4)$ is transitive on the energy-momentum manifold, η_T and ξ_T can represent all the tangent vectors at ζ to the energy-momentum manifold. This ends the proof. ■

8. CONCLUDING REMARKS

S. Sternberg gave a recipe for describing the motion of a classical particle in a Yang-Mills field [8], [14] in association with a principal bundle $G \to P \to M$ with structure group G, G being considered as the internal symmetry group for the classical particle.

A. Weinstein [15] showed that the Sternberg phase space can be obtained by reducing the symplectic manifold $T^*P \times Q, Q$ being a Hamiltonian G-space, through the Weinstein-Marsden reduction procedure associated with the G action.

R. Montgomery [7] proved that the Sternberg symplectic manifold is diffeomorphic with the reduced phase space $J^{-1}(\mu)/G_\mu$ of the standard symplectic manifold (T^*P, σ_P), where P is supposed to be given a connection form, $J : T^*P \to \mathbf{g}^*$ is the moment map associated with the G action, and G_μ is the isotropy subgroup of G at a point μ of $\mathbf{g}^*$.

In the present paper, we have dealt with the case of $P = \dot{\mathbf{R}}^8$, $M = \dot{\mathbf{R}}^5$, and $G = SU(2)$. Thus the $SU(2)$ Kepler problem is a Hamiltonian system for a classical particle in a Yang-Mills field in the sense of Sternberg, while the Yang-Mills field in our case is not defined on S^4, but on $\dot{\mathbf{R}}^5$, which field should be called $BPST$-Yang's monopole field [9], [10], defined on $\dot{\mathbf{R}}^5$ as a generalization of Dirac's monopole field on $\dot{\mathbf{R}}^3$.

The motion of a classical particle associated with the principal bundle $SU(2) \to S^7 \to S^4$ was discussed by Duval and Horváthy [16] and by Guillemin and Uribe [17].

In the rest of this section, we show after M. Kummer [18], [19] that the coadjoint structure discussed in the preceeding section may be generalized.

PROPOSITION 8.1. *Let (M, ω) be a Hamiltonian $G = H \times K$-space, where ω is a symplectic two-form, and H and K are connected Lie groups. Let $\psi_H + \psi_K : M \to \mathbf{h}^* \oplus \mathbf{k}^*$ be the equivariant moment maps associated with the $H \times K$ action, where $\mathbf{h}$ and $\mathbf{k}$ are Lie algebras of H and K, respectively, and $\mathbf{h}^*$ and $\mathbf{k}^*$ are their duals. Assume that $\mu \in \mathbf{k}^*$ is a regular value of ψ_K, and the reduced phase space $\psi_K^{-1}(\mu)/K_\mu$ is a manifold, where K_μ is the isotropy subgroup of K at μ. Assume further that $H \times K_\mu$ acts transitively on $\psi_K^{-1}(\mu)$. Then $\psi_H(\psi_K^{-1}(\mu))$ is a coadjoint orbit $\mathcal{O}$ of H, and $\psi_H : \psi_K^{-1}(\mu) \to \mathcal{O}$ projects to the quotient to define a symplectic covering map $\tilde{\psi}_H : \psi_K^{-1}(\mu)/K_\mu \to \mathcal{O}$, where $\mathcal{O}$ is considered as a symplectic manifold equipped with the Kirillov-Kostant-Souriau form.*

Proof. Since Kummer did not give a complete proof, we here present a proof in brief. For $\xi \in \mathbf{h}$, we denote by ξ_M the infinitesimal generator of the one-parameter group $\exp t\xi$ acting on M. Then the assumption of Hamiltonian G-space provides the moment map ψ_H along with

$$(8.1) \qquad i(\xi_M)\omega = -\langle d\psi_H, \xi \rangle ,$$

where $i()$ and $\langle , \rangle$ denote the interior product and the pairing of $\mathbf{h}^*$ and $\mathbf{h}$, respectively. The equivariance of the moment map $\psi_H + \psi_K : M \to \mathbf{h}^* \oplus \mathbf{k}^*$ implies that, for $h \in H$, $k \in K$, and $p \in M$,

$$\psi_H(hp) = \mathrm{Ad}^*_{h^{-1}}\psi_H(p), \quad \psi_H(kp) = \psi_H(p) , \tag{8.2}$$

$$\psi_K(hp) = \psi_k(p), \quad \psi_K(kp) = \mathrm{Ad}^*_{k^{-1}}\psi_K(p) . \tag{8.3}$$

The transitivity of the $H \times K_\mu$ action on $\psi_K^{-1}(\mu)$ then results in that $\psi_H(\psi_K^{-1}(\mu))$ is a coadjoint orbit $\mathcal{O}$ of H in $\mathbf{h}^*$.

Consider the reduced phase space

$$\pi_\mu : \psi_K^{-1}(\mu) \to \psi_K^{-1}(\mu)/K_\mu ,$$

and define the map

$$\tilde{\psi}_H : \psi_K^{-1}(\mu)/K_\mu \to \mathcal{O}$$

by

$$\tilde{\psi}_H([p]) = \psi_H(p) , \tag{8.4}$$

where $[p]$ denotes the equivalence class of $p \in \psi_K^{-1}(\mu)$, a point of $\psi_K^{-1}(\mu)/K_\mu$. The $\tilde{\psi}_H$ is well defined, of couse. The differential $\tilde{\psi}_{H*}(= d\tilde{\psi}_H)$ of $\tilde{\psi}_H$ then provides

$$\tilde{\psi}_{H*}\left([\xi_M(p)]\right) = \xi_{\mathbf{h}^*}(\psi_H(p)) , \tag{8.5}$$

where $[\xi_M(p)] := \pi_{\mu*}\xi_M(p)$, a tangent vector at $[p]$, and $\xi_{\mathbf{h}^*}$ is the infinitesimal generator of the coadjoint action of $\exp t\xi$ on $\mathbf{h}^*$. According to the Weinstein-Marsden reduction theory [11], on the reduced phase space, there is a unique symplectic form σ determined by $\pi_\mu^*\sigma = i_\mu^*\omega$, where $i_\mu : \psi_K^{-1}(\mu) \to M$ is the inclusion map. Hence, from (8.1) and (8.2), it turns out that

$$\begin{aligned} \sigma\left([\xi_M(p)],[\eta_M(p)]\right) &= -\langle \tilde{\psi}_{H*}([\eta_M(p)]), \xi\rangle \\ &= -\langle \psi_H(p), [\xi, \eta]\rangle . \end{aligned} \tag{8.6}$$

Now suppose that $\tilde{\psi}_{H*}([\eta_M(p)]) = 0$ for some $[\eta_M(p)]$. Then, from (8.6), for every $\xi \in \mathbf{h}$, one has $\sigma([\xi_M(p)],[\eta_M(p)]) = 0$. Since H acts transitively on $\psi_K^{-1}(\mu)/K_\mu$, $[\xi_M(p)]$'s span the tangent space to $\psi_K^{-1}(\mu)/K_\mu$ at $[p]$. The non-degeneracy of σ then means that $[\eta_M(p)] = 0$. Hence $\tilde{\psi}_{H*}$ is injective. Equation

(8.6) also implies that $\tilde{\psi}_H$ is a symplectic map. In fact, since the Kirillov-Kostant-Souriau form κ on $\mathcal{O}$ through $\psi_H(p)$ is defined by

$$\kappa\left(\xi_{\mathbf{h}^*}(\psi_H(p)), \eta_{\mathbf{h}^*}(\psi_H(p))\right) = -\langle \psi_H(p), [\xi, \eta] \rangle , \tag{8.7}$$

Equations (8.5)-(8.7) result in

$$\sigma([\xi_M(p)], [\eta_M(p)]) = \tilde{\psi}_H^* \kappa([\xi_M(p)], [\eta_M(p)]) .$$

Since $[\xi_M(p)]$'s spans the tangent space at every point $[p]$, one has $\sigma = \tilde{\psi}_H^* \kappa$. This ends the proof. ■

For the coadjoint structure we have studied in Sec. 7, we set $H = U(4), K = U(1) \times SU(2)$, and $M = M(2, \mathbb{C}) \times M(2, \mathbb{C})$. The transitivity of the H action on $\psi_K^{-1}(\mu)$ is assured by Prop. 5.1. Hence Prop. 8.1 guarantees that $\psi_K^{-1}(\mu)/K_\mu$ (i.e., $N_k^\sharp/U(1) \times U(1)$ or $N_k^\sharp/SU(2) \times U(1)$) is a symplectic covering of a coadjoint orbit. Our conclusion made in Sec. 7 says that the symplectic covering map is a symplectic diffeomorphism in that case.

It seems likely that the $SU(2)$ bundle $\dot{\mathbf{R}}^8 \to \dot{\mathbf{R}}^5$ is applicable for analyzing the quantized $SU(2)$ Kepler problem. In fact, M. Kibler [20] used it to evaluate the negative eigenvalues for the quantized Kepler problem in five dimensions, which is a special case of the quantized $SU(2)$ Kepler problem.

APPENDIX

Equation (2.10) is written out to give

$$\begin{aligned}
w_0 &= 2(x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3), \\
w_1 &= 2(x_0 y_1 - x_1 y_0 + x_2 y_3 - x_3 y_2), \\
w_2 &= 2(x_0 y_2 - x_2 y_0 + x_3 y_1 - x_1 y_3), \\
w_3 &= 2(x_1 y_2 - x_2 y_1 + x_0 y_3 - x_3 y_0), \\
w_4 &= x_0^2 + x_1^2 + x_2^2 + x_3^2 - y_0^2 - y_1^2 - y_2^2 - y_3^2 .
\end{aligned} \tag{A.1}$$

we take a basis of $su(2)$ as

$$E_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, E_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} . \tag{A.2}$$

The orthogonal system F_a, H_j of vector fields on $\dot{\mathbf{H}}^2$ is given componentwise as follows:

(A.3)

	F_1	F_2	F_3	H_0	H_1	H_2	H_3	H_4
$\partial/\partial x_0$	$-x_1$	$-x_2$	x_3	y_0	y_1	y_2	y_3	x_0
$\partial/\partial x_1$	x_0	$-x_3$	$-x_2$	y_1	$-y_0$	$-y_3$	y_2	x_1
$\partial/\partial x_2$	x_3	x_0	x_1	y_2	y_3	$-y_0$	$-y_1$	x_2
$\partial/\partial x_3$	$-x_2$	x_1	$-x_0$	y_3	$-y_2$	y_1	$-y_0$	x_3
$\partial/\partial y_0$	$-y_1$	$-y_2$	y_3	x_0	$-x_1$	$-x_2$	$-x_3$	$-y_0$
$\partial/\partial y_1$	y_0	$-y_3$	$-y_2$	x_1	x_0	x_3	$-x_2$	$-y_1$
$\partial/\partial y_2$	y_3	y_0	y_1	x_2	$-x_3$	x_0	x_1	$-y_2$
$\partial/\partial y_3$	$-y_2$	y_1	$-y_0$	x_3	x_2	$-x_1$	x_0	$-y_3$

REFERENCES

[1] T.Iwai and Y. Uwano: *The four-dimensional conformal Kepler problem reduces to the three-dimensional Kepler problem with a centrifugal potential and Dirac's monopole field. Classical theory*, J. Math. Phys. **27**, 1523 (1986).

[2] T.Iwai and Y. Uwano: *The quantized MIC-Kepler problem and its symmetry group for negative energies*, J. Phys. A: Math. Gen. **21**, 4083 (1988).

[3] P. Kustaanheimo and E. Stiefel: *Perturbation theory of Kepler motion based on spinor regularization*, J. Reine Angew. Math. **218**, 204 (1965).

[4] E. Stiefel and C. Scheifele: *Linear and Regular Celestial Mechanics*, Springer, Berlin, 1971.

[5] B. Cordani: *Kepler problem with a magnetic monopole*, J. Math. Phys. **27**, 2920 (1986).

[6] T.Iwai: *On a «conformal» Kepler problem and its reduction*, J. Math. Phys. **22**, 1633 (1981).

[7] R. Montgomery: *Canonical formulations of a classical particle in a Yang-Mills field and Wong's equations*, Lett. Math. Phys. **8**, 59 (1984).

[8] S. Sternberg: *Minimal coupling and the symplectic mechanics of a classical particle in the presence of a Yang-Mills field*, Proc. Nat. Acad. Sci. USA, **74**, 5253 (1977).

[9] C.N. Yang: *SU_2 monopole harmonics*, J. Math. Phys. **19**, 320, (1978).

[10] A.A. Belavin, A.M. Polyakov, A.S. Shwartz, and Yu S. Tyupkin: *Pseudoparticle solutions of the Yang-Mills equations*, Phys. Lett. **59B**, 85(1975).

[11] R. Abraham and J.E. Marsden: *Foundations of Mechanics*, Benjamin/Cummings, Reading, MA, 1978, 2nd ed.

[12] S. Kobayashi and K. Nomizu: *Foundations of Differential Geometry*, Interscience, New York, 1963.

[13] J. Moser: *Regularization of Kepler's problem and the averaging method on a manifold*, Comm. Pure Appl. Math. **23**, 609 (1970).

[14] V. Guillemin and S. Sternberg: *Symplectic Techniques in Physics*, Cambridge University Press, Cambridge, 1984.

[15] A. Weinstein: *A universal phase space for particles in Yang-Mills fields*, Lett. Math. Phys., **2**, 417 (1978).

[16] C. DUVAL and P. HORVÁTHY: *Particles with internal structure: the geometry of classical motions and conservation laws*, Ann. Phys. **142**, 10 (1982).
[17] V. GUILLEMIN and A. URIBE: *Clustering theorems with twisted spectra*, Math. Ann. **273**, 479 (1986).
[18] M. KUMMER: *On the three-dimensional Lunar problem and other perturbation problem of the Kepler problem*, Math. Anal. Appl. **93**, 142 (1983).
[19] M. KUMMER: *A group theoretical approach to a certain class of perturbations of the Kepler problem*, Arc. Rat. Mech. Anal. **91**, 55 (1985).
[20] M. KIBLER: *Application of nonbijective transformations to various potentials*, in Lect. Notes in Phys. **313**, Springer, New York (1988).

Manuscript received: May 25, 1990
Revised version: December 7, 1990